Roll No. | P | S | | | | | | |

3115124

DOON UNIVERSITY, DEHRADUN
End Semester Examination, May 2024
School of Physical Sciences
B.Sc.(Mathematics)IV-Semester
Course: MAC-252: Riemann Integration and Series of Functions

Time Allowed: 2 Hours **Maximum Marks: 100**

Note: *Make sure to write your roll number at top left of the question paper. Attempt* *__any four__* *questions from section A, section B and* *__any two__* *questions from section C. The weightage of paper will be 50% of total marks obtained in the semester.*

Section: A ($4 \times 5 = 20$ Marks)

(1) Find the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{5^k}{k} x^k$.

(2) Explain point wise and uniform convergence of a sequence $\{f_n(x)\}, x \in E$ of functions.

(3) Provide an example of sequence of continuous functions converging point wise to a discontinuous function.

(4) Find the domain where the power series $\sum_{n=1}^{\infty} \left(\frac{k+1}{k}\right)^{2k^2} x^k$ converges.

(5) Show that the radius of convergence R of the power series $\sum_{k=1}^{\infty}$ is determined by $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$.

Section: B ($10 \times 4 = 40$ Marks)

(6) Apply any test to discuss convergence/diveregence of the improper integrals

$$\int_0^{\infty} (1 + x^{10})^{-1/5}\, \mathrm{d}x.$$

(7) Consider the series $\sum a_k$, where $a_k > 0$ for all k. Let $L = \limsup \sqrt[k]{a_k}$. Then prove that the series $\sum a_k$ converges if $L < 1$.

(8) Show that the sequence $f_n(x) = 1/(x + n)$ converges uniformly on $[0, \infty)$ to $f(x) \equiv 0$.

(9) Show that the sequence $\{f_n(x)\}$, $f_n(x) = x^n$ does not converge uniformly to $f(x) \equiv 0$ on $[0, 1]$.

(10) Discuss the convergence of the sequence $\{f_n(x)\}$, $f_n(x) = x^n(1 - x)$ on $[-1, 1]$. Also plot the functions f_1, f_2 and f_3 to visualize the situation.

Section: C ($20 \times 2 = 40$ Marks)

(11) (a) Give an example to show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx.$$

Under what conditions the equality holds ?

(b) Test the series $\sum_{k=0}^{\infty}(3 + \sin(k\pi/2))(x - 1)^k$ for convergence.

(12) (a) Consider the series $\sum a_k$, where $a_k > 0$ for all k. Let $L = \limsup \dfrac{a_{k+1}}{a_k}$. Then prove that the series $\sum a_k$ converges if $L < 1$.

(b) Find the interval of convergence of the series $\sum_{k=1}^{\infty}(3 - \cos k\pi)x^k$. Which test provides better result in this problem ?

(13) Let $\{f_n(x)\}$ be a sequence of functions continuous on a set E and $f_n \to f$ uniformly. Then, prove that $f(x)$ is continuous. Also explain why the sequence $\{f_n(x)\}$, given by

$$f_n(x) = \begin{cases} 1 - nx, & \text{if } 0 \leq x \leq 1/n, \\ 0, & \text{if } 1/n < x \leq 1, \end{cases}$$

is not uniformly convergent.