1-6-2024

DOON UNIVERSITY, DEHRADUN

End Semester Examination, II Semester, 2024

Academic Year 2023-24 (Even Semester)

School of Physical Science Department Name: Mathematics

Programme Name: Integrated M.Sc.

Course Code with Title: MAC-153, Group Theory-1

Time Allowed 2.00 Hours ***Maximum Marks: 50***

SECTION: A

I. Attempt all questions: **(10 marks)**

a. In a finite group G, if H is the only subgroup of its order, then prove that H is a normal subgroup of G. **(2)**

b. Find the order of (8,4,10) in the group $Z_{12}\oplus Z_{60}\oplus Z_{24}$. **(1)**

c. Find the maximum possible order of any element in S_{15} **(1)**

d. Find the order of $3+<6>$ in $Z_{18}/<6>$. **(1)**

e. Let $G = (R,+). Define\ \emptyset: G \to G\ by\ \emptyset(x) = x^3, \forall x \in G$. Check if $\emptyset$ is a homomorphism. Justify your answer. **(2)**

f. Define the following terms:

i. Normalizer of an element of a group G.

ii. External direct product of a finite number of groups. **(1)**

g. if G is a group and H is a subgroup of index 2 in G, then prove that H is a normal subgroup of G. **(2)**

SECTION: B

II. Attempt any five: **(5X4=20 marks)**

a. prove that centre of a group is a normal subgroup of the group.

b. What is an Alternating group. Show that $A_n \leq S_n$.

c. Describe the structure of S_5. For the following permutations σ & τ $\in S_9$, write $\sigma\tau$ as the product of disjoint cycles:

$$\sigma = (\mathbf{23415})(4768)\ ;\ \tau = (\mathbf{4397})(\mathbf{15274})$$

d. If in a group G, $a^5 = e, aba^{-1} = b^2$ for $a, b \in G$ then show that O(b)=31.

e. Let N be a normal subgroup G and H a subgroup of G. Then show that NH is a subgroup of G.

f. Show that the additive group Z of all integers is homomorphic to the multiplicative group G = {1,-1}

SECTION: C

III. **Attempt any two:** **2X10=20**

a. **(i).** Prove that the external direct product of a finite number of groups is a group under component-wise product. Find the order and inverse (2,3) in $Z_3 \oplus Z_5$. Is $Z_3 \oplus Z_6$ is cyclic? justify your answer. **(7)**

(ii). f: $(G, +) \rightarrow (G', *), f(x) = 2^x$. Show that f is a homomorphism. **(3)**

b. **(i).** Prove that a homomorphism $\emptyset$ of a group is injective iff $ker\emptyset = < e >$. **(6)**

(ii). Determine all homomorphism from Z_{20} to Z_8. **(4)**

c. **(i).** Define factor group. Let N∆G and H< G, containing N. then show that $H/_N < G/_N$. **(5)**

(ii) Draw the Cayley table for $S_3/_{\langle(123)\rangle}$. **(5)**