22-3-2018

DOON UNIVERSITY, DEHRADUN

Mid Semester Examination, 2017-2018

School of Physical Sciences

M.Sc.(Mathematics)II-Semester

Course: MAC-451: Functional Analysis

Time Allowed: 2 Hours **Maximum Marks: 30**

Note:

Attempt all questions from Section A, any four questions from Section B and any two questions from Section C.

Section: A ($1.5 \times 4 = 6$ Marks)

(1) Compute the norm of the operator $T : C[0,1] \to C[0,1]$ defined by

$$Tx = \int_0^1 x(t)\, \mathrm{d}t.$$

(2) What additional properties are attains by a metric that is induced by a norm ?

(3) Explain, why the space l^p is not an inner product space in case $p \neq 2$.

(4) Provide an example that implies that a metric may not be induced by a norm.

Section: B ($3 \times 4 = 12$ Marks)

(1) In an inner product space $x_n \longrightarrow x$, $y_n \longrightarrow y$ imply that $\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$. What is the implication of this result ?

(2) Consider $C[0, 2\pi]$ and determine the smallest r such that $y \in \widetilde{B}(x;r)$, where $x(t) = \sin t$ and $y(t) = \cos t$.

(3) Let $T : D(T) \longrightarrow Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces. Then T is continuous on $D(T)$ if T is continuous at a single point.

(4) Establish the Schwarz inequality

$$|\langle x, y \rangle| \leqslant \|x\| \|y\|.$$

(5) Prove that a projection operator P on a Hilbert space $P : H \to Y$, Y closed maps Y onto Y.

Section: C ($6 \times 2 = 12$ Marks)

(1) (a) Prove the Hölder inequality

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |\eta_j|^q \right)^{1/q},$$

where $p > 1$ and $1/p + 1/q = 1$.

(b) Show that the closed unit ball

$$B(0;1) = \{x \in X : \|x\| \leqslant 1\}$$

in a normed space X is convex.

(2) Prove that if a normed space X has the property that the closed unit ball $M = \{x : \|x\| \leqslant 1\}$ is compact, then X is finite dimensional.

(3) Let $\{x_1, x_2, ..., x_n\}$ be a linearly independent set of vectors in a normed space X. Show that, for every choice of scalers α_i, $i = 1, 2, ..., n$ there exists a constant K such that

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geqslant K\big(|\alpha_1| + \cdots + |\alpha_n|\big).$$