16/12/17

DOON UNIVERSITY, DEHRADUN

End Semester Examination, Second Semester, 2017

M.Sc.(Mathematics)

Course: MAC-412: Applied Functional Analysis

Time Allowed: 3 Hours **Maximum Marks: 100**

Note: *Attempt all questions from section A, any four questions from section B and any two questions from section C.*

Section: A (5×4 = 20 Marks)

Prove following propositions:

(1) A contraction T on a metric space X is a continuous mapping.

(2) Does f defined by $f(t,x) = |x|^{1/2}$ satisfy a Lipschitz condition on $[0,1]$?

(3) Strong convergence implies weak convergence with the same limit but the converse is not generally true.

(4) If (x_n) and (y_n) are sequences in the same normed space X, then $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ implies $x_n + y_n \xrightarrow{w} x + y$ as well as $ax_n \xrightarrow{w} ax$, where a is any scalar.

Section: B (10×4 = 40 Marks)

(1) Let (x_n) be a sequence in a normed space X. Prove that if $\dim X < \infty$, then weak convergence implies strong convergence.

(2) Consider a metric space (X, d), where $X \neq \phi$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.

(3) In analysis, a usual sufficient condition for the convergence of an iteration $x_n = g(x_{n-1})$ is that g be continuously differentiable and

$$|g'(x)| \leqslant \alpha < 1.$$

Verify this by the use of Banach's fixed point theorem.

(4) Solve by iteration, choosing $x_0 = 1$:

$$x(t) - \mu \int_0^1 x(\tau)\, \mathrm{d}\tau = 1, \qquad |\mu| < 1.$$

(5) Prove that $x(t) - \mu \int_a^t k(t,\tau)x(\tau)\, \mathrm{d}\tau = v(t)$ has a unique solution for every μ if $v \in C[a,b]$ and k is continuous on $a \leqslant \tau \leqslant t$, $a \leqslant t \leqslant b$.

Section: C (20×2 = 40 Marks)

(1) Let $x_n \xrightarrow{w} x$. Then:

(a) The weak limit x of (x_n) is unique.

(b) Every subsequence of (x_n) converges weakly to x.

(2) If an $A-$Summability method with matrix $A = (a_{n,k})$ is regular then,

(a) $\lim_{n\to\infty} a_{n,k} = 0$,

(b) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$ and
(c) $\sum_{k=1}^{\infty} |a_{n,k}| \leq \gamma$.

(3) (a) If a system

$$x = Cx + b \qquad (C = (c_{jk}), b \text{ given})$$

of n linear equations in n unknowns $\xi_1, \xi_2, ..., \xi_n$ (the components of x) satisfies

$$\sum_{k=1}^{n} |c_{jk}| < 1 \qquad (j = 1, ..., n),$$

it has precisely one solution x.

(b) Apply Cesáro's summability method(Averaging method) to the following sequences

$(i)\ (1, 0, 1, 0, 1, \cdots)$ $\qquad$ $(ii)\ \left(1, 0, -\frac{1}{4}, -\frac{2}{8}, -\frac{3}{16}, \cdots\right)$.