20-3-17

DOON UNIVERSITY, DEHRADUN

Mid Semester Examination, 2016-2017

School of Physical Sciences

M.Sc.(Mathematics)II-Semester

Course: MAC-451: Functional Analysis

Time Allowed: 2 Hours **Maximum Marks: 30**

Note:

Attempt all questions from Section A, any four questions from Section B and any two questions from Section C.

Section: A (6/5 × 5 = 6 Marks)

(1) Let T be a bounded linear operator, then the norm can be defined by

$$\|T\| = \sup_{x \in D(T); \|x\|=1} \|Tx\|.$$

(2) Prove that the real line R is separable.

(3) If a metric is induced by a norm, what additional properties it attains other than those of a metric ?

(4) Let α and β be any positive numbers and $p > 1, q$ be such that $1/p + 1/q = 1$. Show that

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

(5) Prove that for any two bounded linear operators T_1 and T_2 on a normed space, there holds

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|.$$

(6) Prove that the space $C[a, b]$ is not an inner product space.

Section: B (3 × 4 = 12 Marks)

Prove following prepositions:

(1) Let $X = (X, d)$ be a metric space. Then:
  (a) A convergent sequence in X is bounded and its limit is unique.
  (b) $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X, then $d(x_n, y_n) \longrightarrow d(x, y)$.

(2) In an inner product space $x_n \longrightarrow x$, $y_n \longrightarrow y$ imply that $\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$. What is the implication of this result ?

(3) Consider $C[0, 2\pi]$ and determine the smallest r such that $y \in \widetilde{B}(x; r)$, where $x(t) = \sin t$ and $y(t) = \cos t$.

(4) Let $T : D(T) \longrightarrow Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces. Then T is continuous if and only if T is bounded.

(5) On a finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.

Section: C (6 × 2 = 12 Marks)

(1) Prove the Hölder inequality

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |\eta_j|^q \right)^{1/q},$$

where $p > 1$ and $1/p + 1/q = 1$.

(2) The class of continuous functions, $C[a, b]$ is complete.

(3) The space l^∞ is not separable.