8-12-16

DOON UNIVERSITY, DEHRADUN

End Semester Examination, First Semester, 2016-17

School of Physical Sciences

Class: Integrated M.Sc. Mathematics **Semester: II**

Course: Theory of Real Functions **Course Code: MAC-105**

Time Allowed: 3Hours ***Maximum Marks: 100***

Note: Attempt all six questions in Section A. Each question carries 3 marks.

Attempt any six questions in Section B. Each question carries 7 marks.

Attempt any four questions in Section C. Each question carries 10 marks.

SECTION: A

(Very Short Answer Type Questions)

(Marks: 6X3=18)

1. Define the following terms:
   (a) ε-δ definition of continuity
   (b) uniform continuity
   (c) left and right hand limits.
2. If a function f is continuous on a closed and bounded interval $[a, b]$, then prove that it is bounded in $[a, b]$.
3. Prove that continuity is a necessary condition but not a sufficient condition for the existence of a finite derivative.
4. State Cauchy's mean value theorem and verify it for $f(x) = \sin x$, $g(x) = \cos x$ in $[-\pi/2, 0]$.
5. Find the values a, b, c so that $\lim_{x\to 0} \frac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}$ may be equal to 2.
6. Find the maximum and minimum values of the function
   $f(x) = 12x^5 - 45x^4 + 40x^3 + 6,\ x \in R.$

SECTION: B

(Short Answer Type Questions)

(Marks: 6X7=42)

7. State and prove sequential criterion for continuity.
8. If $f: (0, \infty) \to R$ is a function defined by $f(x) = \frac{1}{x}$, prove that f is uniformly continuous on $[a, \infty)$ where $a > 0$. Show that f is continuous but not uniformly on $(0,\ \infty)$.
9. The function f defined by $f(x) = \begin{cases} x^2 + 3x + a, & if\ x \le 1 \\ bx + 2, & if\ x > 1 \end{cases}$ is given to be differentiable for every x. Find a and b.
10. State and prove Rolle's theorem.
11. Assuming the validity of expansion, show that $\sin(e^x - 1) = x + \frac{x^2}{2!} + \frac{5x^{43}}{4!} + \cdots$
12. If $0 < x < 1$, show that $2x < \log\frac{1+x}{1-x} < 2x\left(1 + \frac{x^2}{3(1-x^2)}\right)$.
13. Let I be an open interval and let $f: I \to R$ have a second derivative on I, then prove that f is convex if and only if $f''(x) \ge 0\ \forall x \in I$.

SECTION: C
(Long Answer Type Questions)

(Marks: 4X10=40)

14. Discuss the classification of discontinuities of a function with suitable examples.

15. State and prove (a) Caratheodory's and (b) Darboux's theorems.

16. ~~State and prove Lagrange's Mean value theorem and examine its validity for the~~ function $f(x) = x \sin x$ in the interval $\left[0, \frac{\pi}{2}\right]$.

17. State and prove Taylor's theorem with Lagrange's form of remainder. Also, show that $x - \frac{x^3}{3!} \leq \sin x \leq x,\ x \geq 0.$

18. State and prove the necessary and sufficient conditions for the existence of maxima and minima of a function based on its first derivative.